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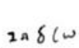
EE123
MT1

- $\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases} = u[n]-u[n-1]$
- $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} = \sum_{k=0}^{\infty} \delta[n-k]$
- memoryless - only current $x[n]$
- linear - $ax_1 + bx_2 \leftrightarrow ay_1 + by_2$
- time invariant - $x[n-n_0] \leftrightarrow y[n-n_0]$
- causal - only dep on present/past
- BIBO stability
- LTI systems
  - $y[n] = h[n] * x[n]$; $Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$
  - stability - $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$
  - linear constant coefficient difference equations
    $\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$
- DTFT
  - $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
  - $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$
    exists if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ (absolutely summable)
  - Ideal LPF ω_c $h[n] = \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi} \text{sinc}(\frac{\omega_c}{\pi} n)$
  - Properties $x[n] \leftrightarrow X(e^{j\omega})$
    - $x^*[n] \leftrightarrow X^*(e^{-j\omega})$
    - $x^*[-n] \leftrightarrow X^*(e^{j\omega})$
    - real $x[n]$
      - $X(e^{j\omega}) = X^*(e^{-j\omega})$
    - $ax[n] + by[n] \leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$
    - $x[n-n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$
    - $e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$
    - $x[-n] \leftrightarrow X(e^{-j\omega})$ // $X^*(e^{j\omega})$ if x real
    - $n x[n] \leftrightarrow j \frac{dX(e^{j\omega})}{d\omega}$
    - $x[n] * y[n] \leftrightarrow X(e^{j\omega}) Y(e^{j\omega})$
    - $x[n] y[n] \leftrightarrow \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta$
  - Parseval's Thm
    - $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$
    - $\sum_{n=-\infty}^{\infty} x[n] y^*[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) Y^*(e^{j\omega}) d\omega$
- Common Transforms
  - $\delta[n] \leftrightarrow 1$
  - $\delta[n-n_0] \leftrightarrow e^{-j\omega n_0}$
  - $1 \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi \delta(\omega + 2\pi k)$
  - $a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}}$ $(|a|<1)$
  - $u[n] \leftrightarrow \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega + 2\pi k)$
  - $(n+1) a^n u[n] \leftrightarrow \frac{1}{(1-ae^{-j\omega})^2}$ $(|a|<1)$
  - $\frac{r^n \sin(\omega_p(n+1))}{\sin(\omega_p)} u[n] \leftrightarrow \frac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$ $(|r|<1)$
  - $\frac{\sin(\omega_c n)}{\pi n} \leftrightarrow$  $(2\pi$ periodic$)$
  - $x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{o/w} \end{cases} \leftrightarrow \frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)} e^{-j\omega M/2}$
  - $e^{j\omega_0 n} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 + 2\pi k)$
  - $\cos(\omega_0 n + \phi) \leftrightarrow \sum_{k=-\infty}^{\infty} (\pi e^{j\phi} \delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi} \delta(\omega + \omega_0 + 2\pi k))$
- Geometric Series
  $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$; $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$; $\sum_{k=m}^{n} ar^k = \frac{a(r^m - r^{n+1})}{1-r}$
- $\text{sinc}(n) = \frac{\sin(\pi n)}{\pi n}$

- Z-transform
  - $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$
  - $z = re^{j\omega}$ (if $r=1 \Rightarrow$ DTFT)
  - ROC $\sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty$
    - always circle, disk, etc
    - right sided $r >$ 
    - left sided $r <$ 
  - Common Transforms
    - $\delta[n] \leftrightarrow 1$ // all r
    - $u[n] \leftrightarrow \frac{1}{1-z^{-1}}$ $|z|>1$
    - $-u[-n-1] \leftrightarrow \frac{1}{1-z^{-1}}$ $|z|<1$
    - $\delta[n-m] \leftrightarrow z^{-m}$ $z \neq 0, z \neq \infty$
    - $a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}$ $|z|>|a|$
    - $-a^n u[-n-1] \leftrightarrow \frac{1}{1-az^{-1}}$ $|z|<|a|$
    - $na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}$ $|z|>|a|$
    - $-na^n u[-n-1] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}$ $|z|<|a|$
    - $\cos(\omega_0 n) u[n] \leftrightarrow \frac{1-\cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}}$ $|z|>1$
    - $\sin(\omega_0 n) u[n] \leftrightarrow \frac{\sin(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}}$ $|z|>1$
    - $r^n \cos(\omega_0 n) u[n] \leftrightarrow \frac{1 - r\cos(\omega_0) z^{-1}}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}}$ $|z|>r$
    - $r^n \sin(\omega_0 n) u[n] \leftrightarrow \frac{r\sin(\omega_0) z^{-1}}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}}$ $|z|>r$
    - $\begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{o/w} \end{cases} \leftrightarrow \frac{1 - a^N z^{-N}}{1 - az^{-1}}$ $|z|>0$
  - ROC Properties
    - circle/ring
    - FT converges if unit circle
    - ROC no poles
    - x finite, ROC all (except z=0 or ∞ maybe)
    - connected
  - stable ⟺ ROC has unit circle
  - Inverse Z transform
    - inspection
    - Partial fraction expansion
    - Power series expansion - good for delta or z^{-n} coeff
  - Properties
    - $ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z)$ $R_{x_1} \cap R_{x_2}$
    - $x[n-n_0] \leftrightarrow z^{-n_0} X(z)$ R_x
    - $z_0^n x[n] \leftrightarrow X(z/z_0)$ $|z_0| R_x$
    - $n x[n] \leftrightarrow -z \frac{dX(z)}{dz}$ R_x
    - $x^*[n] \leftrightarrow X^*(z^*)$ R_x
    - $x^*[-n] \leftrightarrow X^*(1/z^*)$ $1/R_x$
    - $x_1[n] * x_2[n] \leftrightarrow X_1(z) X_2(z)$ $R_{x_1} \cap R_{x_2}$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

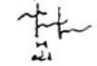
$$\Updownarrow$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- DFT
  - $W_N = e^{-j\frac{2\pi}{N}}$
  - $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$
  - $x[n] = \sum_{k=0}^{N-1} X[k] W_N^{-kn}$
  - $x_1[n] \circledast x_2[n] = \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$
    - circular convolution
  - $\text{DFT}^{-1}\{X[k]\} = \frac{1}{N}(\text{DFT}\{X^*[k]\})^*$
  - Properties
    $X[n] \leftrightarrow N x[((-k))_N]$
    $x[((n-m))_N] \leftrightarrow W_N^{km} X[k]$
    $W_N^{-ln} x[n] \leftrightarrow X[((k-l))_N]$
    $x_1[n] \circledast x_2[n] \leftrightarrow X_1[k] X_2[k]$
    $x_1[n] x_2[n] \leftrightarrow \frac{1}{N}(X_1[k] \circledast X_2[k])$
    $x^*[n] \leftrightarrow X^*[((-k))_N]$
    $x^*[((-n))_N] \leftrightarrow X^*[k]$
    $X[k] = X^*[((-k))_N]$ if x real

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

  - Linear Convolution w/ DFT
    - $x[n]$ length L, $h[n]$ length P, $x[n] * h[n]$ len $L+P-1$
    - pad 0s to $L+P-1$, proceed normally
  - overlap add
    - split separate sequences, add last P-1 to next output



  - overlap save
    - overlapping sequences, toss first couple, save and output last L.
  - FFT
    - Decimation in Time
      - idea: split $x[n]$ into diff sequence even odd
      - idea: butterfly base case
      - ends up being bit reverse order
    - Decimation in Frequency
      - idea: split $X[k]$ into even odd
  - note LPF on DFT lots of artifacts 
    - only 0 guaranteed at k freq
  - Windowing Effect
    - rectangular - small main lobe, huge side lobes
    - Hann - beta value, ↑ ⇒ big main, small side
    - ↑ length ⇒ smaller main lobe
    - zero padding adds no info
  - Time Dependent FT (DFT) (STFT)
    - $X[n,\omega) = \sum_{m=-\infty}^{\infty} x[n+m] \underset{\text{window}}{w[m]} e^{-j\omega m}$
    - $X_r[k] = \sum_{m=0}^{L-1} x[rR+m] w[m] W_N^{km}$
  - Heisenberg Box - time freq tradeoff $\sigma_t \sigma_f \geq 1/2$
  - Wavelets
    - $Wf(u,s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \psi^*(\frac{t-u}{s}) dt$
    - ↑ s wider lower freq
    - ψ mother wavelet $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$ unit norm
      $\int_{-\infty}^{\infty} \psi(t) = 0$ bandpass
    - $\psi_{i,n}(t) = \frac{1}{\sqrt{2^i}} \psi(\frac{t - 2^i n}{2^i})$
      scale shift
    - real-useful, ex denoise piecewise signals w/ Haar

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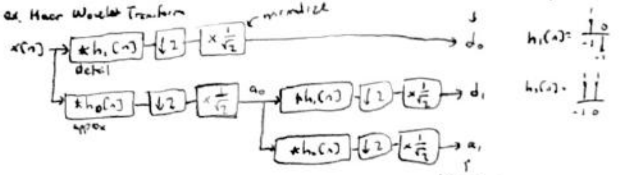
MT2

- Fast Haar Wavelet (Fast DWT) w/ Filter Banks
  - idea: split into 2, approximation - LPF, detail - HPF $O(N)$!
  - keep iterating on approx to get components
- ex. Haar Wavelet Transform



order from $a \to d$
low → high

- ex. Haar Wavelet Inverse Transform
  - same as above but reverse all arrows, switch $\downarrow 2$ to $\uparrow 2$
  - switch $h_1[n]$ to $g_1[n]$ and $h_0[n]$ to $g_0[n]$

- Sampling
  - $x_c(t) \to \boxed{C/D} \to x[n] = x_c(nT)$
    - $x_s(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT)$ impulse train equivalent
    - $X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$ } $\omega = \Omega T$
    - $X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n}$
    - $s(t) = \sum_n \delta(t-nT) \Leftrightarrow S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$
    - $X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$, $\Omega_s = \frac{2\pi}{T}$
      - scaled and repeated every $\frac{2\pi}{T}$
      - careful of aliasing

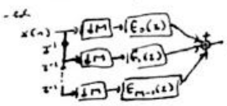
- Reconstruction
  - Nyquist Sampling Theorem
    - if $\Omega_s > 2\Omega_N$, then can uniquely reconstruct (worst case)
      - Ω_N: max freq (bandlimited)
    - Nyquist Rate: sample rate ($2\times$ max freq)
    - Nyquist Frequency: max freq in signal
  - idea: impulse train, LPF scaled by T (sinc!) = 0 every T
    - $x_r(t) = \sum_n x[n] \,\text{sinc}\left(\frac{t-nT}{T}\right)$
    - $x[n] \to \boxed{D/C} \to x_r(t)$, T

- DT Processing
  - $x_c(t) \to \boxed{C/D}_{T_1} \to \boxed{H(e^{j\omega})} \to \boxed{D/C}_{T_2} \to y_r(t)$
  - if $h[n]$ is LTI, then system is LTI if bandlimited
  - ex. resampling is just sinc interpolation!

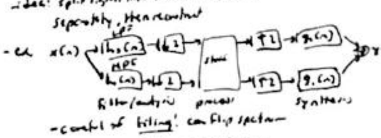
- Resampling
  - Downsampling
    - idea: like sampling cont time
    - $x[n] \to \boxed{\text{LPF } \pi/M} \xrightarrow{\tilde{x}[n]} \boxed{\downarrow M} \to \tilde{x}[nM]$ ← equal $\frac{1}{MT}$, which is right!
    - DTFT - stretched, scaled by $\frac{1}{M}$, tiled
  - Upsampling
    - sinc interpolation; add 0's then convolve (LPF!)
    - $x[n] \to \boxed{\uparrow L} \to \boxed{\text{LPF, gain } L, \omega_c = \pi/L} \to$
      $\omega = \pi \to \omega = \frac{\pi}{L}$
    - DTFT - compress and tile, so need to scale and $\frac{L}{T}$
  - Resample by rational $T' = \frac{TM}{L}$ ← down, ← up
    - $x[n] \to \boxed{\uparrow L} \to \boxed{\text{LPF, gain } L, \omega_c = \min(\frac{\pi}{L}, \frac{\pi}{M})} \to \boxed{\downarrow M}$
    - issue: too slow, hard to make good filter if like 1.01

- Filter Design
  - affects phase and magnitude
  - usually window to get M^{th} order, $0 \le n \le M$ (length $M+1$)
    - length - affects transition width
    - type - affect transition width / ripple
    - can modulate to shift response, ex. make LPF → BPF, HPF ($2\cos(\omega_c n)$, $(-1)^n$) } windowing method
  - TBW (Time Bandwidth Product)
    - $T(BW) = (M+1)\frac{\omega}{2\pi} = $ # zero crossings (include ends)
      big = more sinc-like ω: bandwidth (out of 2π)
  - Optimal Design
    - specify some criteria, ex. don't care areas
    - ex. weighted least squares $\int_{-\pi}^{\pi} W(\omega) |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$ (ideal) } optimal
    - Chebyshev Design - min max $|H(e^{j\omega}) - H_d(e^{j\omega})|$
      - Parks-McClellan Alg. Remez exchange alg., convex optimization

- Polyphase Decomposition
  - ex. good for more efficient resampling
  - idea: split impulse response
  - Interchanging Operations
    $\to \boxed{H(z)} \to \boxed{\uparrow L} \to \equiv \to \boxed{\uparrow L} \to \boxed{H(z^L)} \to$
    $\to \boxed{\downarrow M} \to \boxed{H(z)} \equiv \to \boxed{H(z^M)} \to \boxed{\downarrow M} \to$
    Can also do $H(z) \Rightarrow H(z^{1/L})$, $\Rightarrow H(z^{1/M})$
    iff it's rational, basically means it got zeros between L, M points
  - define $h[n] = \sum_{k=0}^{M-1} h_k[n-k]$ (every M sample, start at k)
  - $H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}$ (delay)
    $e_k[n] = h_k[nM]$
  - allows us to swap filter and decimator
  - ex.



- Multirate Filter Bank
  - idea: split signal into subbands, to process separately, then reconstruct
  - ex.



  - careful of tiling! can flip spectrum
  - can continue to decompose further

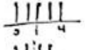
- Rational System Response
  $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$
  $H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$ (zero, pole)
  - Magnitude Response
    - product of magnitudes, note $|1 - d_k e^{-j\omega}| = |e^{j\omega} - d_k|$ vector
  - Phase Response
    - $\arg(H(e^{j\omega}))$ - unwrapped phase
    - $ARG(H(e^{j\omega}))$ - wrapped phase $(-\pi, \pi)$
    - $grd(H(e^{j\omega})) = -\frac{d}{d\omega}(\arg(H(e^{j\omega})))$
      - group delay
      - if nonlinear, diff freq diff delay!
    - additive at angles

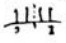
- idea: discretize ideal freq response
  - P_{points} $P \gg M+1$ ($P = 15M$ good)
- first get non-causal then shift by $M/2$ to make causal
- ex. least squares use matrix that's like IDFT
  but only put in points u care about
  $A\vec{h} = \vec{b} \Rightarrow \vec{h} = (A^* A)^{-1} A^* b$ (be symmetric, remove A^T)
- ex. GLP filter
  - remember $H(e^{j\omega}) = h[0] + 2\cos(\omega)h[1] + 2\cos(2\omega)h[2] + \dots + 2\cos(\frac{M}{2}\omega)h[\frac{M}{2}]$ } least squares
- can also weight it
  $\vec{h} = (A^* W^2 A)^{-1} W^2 A^* \vec{b}$

- what about ripple?
  - like ok w/ ripple = & in between ok
  - becomes linear program
  - use convex optimization like CVX } convex opt

- Digital Modulation Schemes
  - On-off keying (OOK) - on or off
  - Phase-shift keying (PSK) - choose phase
  - Quadrature Amplitude Modulation (QAM) - phase + magnitude
  - Frequency Shift Keying (FSK) - choose freq
    - Minimum Shift Keying (MSK) - FSK w/ half cycle diff, smooth?
- note FM is frequency mod, so big freq == big amplitude
- Gaussian noise == all freqs in expectation
- Considerations
  - Pulse-shaping - smaller sidelobes / leakage
  - Phase discontinuity = big sidelobes
    - want same phase for each bit
  - eye diagram - gives best sample point
  - bit error rate (BER)
  - PLL - get optimal sample, idea: sample at between zero crossings (careful jitter)

- All Pass System ($H_{ap}(z)$)
  - $H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$ pole zero conjugates at inverse
  - $|H(e^{j\omega})| = 1 \; \forall \omega$, $\angle H(e^{j\omega})$ not constant
  - properties: $grd(H(e^{j\omega})) > 0$ → causal
    if stable, $\arg(H(e^{j\omega})) \le 0$ → phase lag
- Minimum Phase System ($H_{min}(z)$)
  - $\underbrace{\text{stable + causal}}_{\text{poles in unit circle}}$ $H(z)$ whose inverse also $\underbrace{\text{stable + causal}}_{\text{zeros in unit circle}}$ ($\frac{1}{H(z)}$)
    - min group delay
    - min energy delay
- All Pass Min Phase Decomposition
  - stable causal $H(z) = H_{min}(z) \cdot H_{ap}(z)$
  - Algorithm
    1) Get $H_{ap}(z)$ using zeros/poles outside
    2) $H_{min}(z) = \frac{H(z)}{H_{ap}(z)}$ // basically place pole/zero at inverse of zero
  - allows compensation for min phase part
  - max phase - all zeros outside, causal, unstable inverse
- Generalized Linear Phase (GLP) systems
  - $H(e^{j\omega}) = \underbrace{A(e^{j\omega})}_{\text{real}} e^{-j\alpha\omega + j\beta}$
  - $grd(H(e^{j\omega})) = \alpha$ (except at sign changes)
    - constant group delay! of $M/2$!
  - FIR GLP - symmetry $h[n] = h[M-n]$ (length $M+1$, so $M+1$ taps)





    - Type I (M even) } $h[n] = h[M-n]$
    - Type II (M odd)
    - Type III (M even) } $h[n] = -h[M-n]$
    - Type IV (M odd)
  - Transfer Function Real Part
    - Type I $A(e^{j\omega}) = h[\frac{M}{2}] + 2\sum_{k=1}^{M/2} h[\frac{M}{2} - k]\cos(\omega k)$
    - Type II $A(e^{j\omega}) = \sum_{k=1}^{(M+1)/2} 2h[\frac{(M+1)}{2} - k]\cos(\omega(k - \frac{1}{2}))$
      - note $\omega = \pi$, always 0, never HPF
    - Type III $A(e^{j\omega}) = j \cdot 2\sum_{k=1}^{M/2} h[\frac{M}{2} - k]\sin(\omega k)$
      - note $\omega = 0, \pi$, both always 0, always BPF
    - Type IV $A(e^{j\omega}) = j \cdot 2\sum_{k=1}^{(M+1)/2} h[\frac{(M+1)}{2} - k]\sin(\omega(k - \frac{1}{2}))$
      - note $\omega = 0$ always 0, never LPF
  - if $a = re^{j\theta}$ is zero $\frac{1}{a^*}$ also zero
  - FIR == no poles (actually all poles at origin)
  - $H(z) = H_{min}(z) H_{max}(z) H_{uc}(z)$
    <1, >1, $=1$ (on unit circle)

- Image/Video Compression
  - throw out data don't need
  - JPEG2000 pretty cool
  - exploit time, motion vectors
  - PCA/SVD
- Compressive Sampling
  - sparsity good for denoising
  - measure + compress + process all in one
  - < Nyquist sample rate

- Practical ADC/DAC
  - sharp analog filter hard, so use simple and oversample → downsample to reduce noise, sharp digital filter instead
  - quantization
    - X_m: dynamic range
    - B bits, $\Delta = \frac{X_m}{2^B}$, B+1 bits
    - model error as noise
      bounded $-\Delta/2 < e < \Delta/2$
    - if assume random, works well
      $\sigma_e^2 = \frac{\Delta^2}{12}$
  - $SNR = 6.02B + 10.8 - 20\log_{10}(\frac{X_m}{\sigma_x}) + 10\log_{10} M$ (dB, oversampling trick)
  - DAC
    - usually ZOH, not perfect
    - reconstruction filter compensate
    - practically,
      - digital upsample to compensate