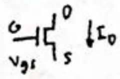
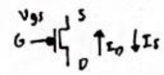
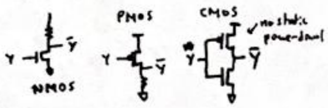
- CMOS (Complimentary Metal Oxide Semiconductor)
  - NMOS



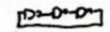
    0 - open
    1 - closed
  - PMOS



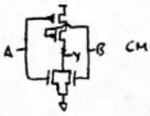
    0 - closed
    1 - open
- logic gates
  - Inverter



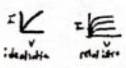
  - NAND
  - ring oscillator

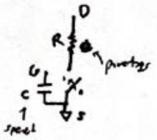


    odd prime #
  - NOR



    CMOS
- Transistor Model



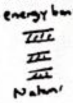


- $Q = CV$ Capacitance

  $I = C\frac{dV}{dt}$ $E = \frac{1}{2}CV^2$

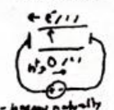
  $V = V_0 e^{-t/RC}$ $P_{dissipated} = CV^2 f$ ← the half from resistor

  $V = V_{DD}(1 - e^{-t/RC})$
- Semiconductors (Silicon)
  - energy band

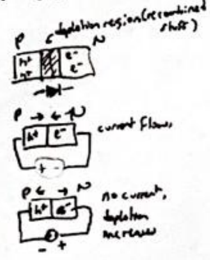


    //// conduction band (empty)

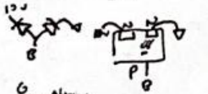
    //// valence band (full)
  - hole current

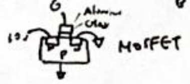


    - happen naturally thru thermal excitation
  - P-type - h^+ extra
  - N-type - e^- extra
  - Diode



  - Transistor



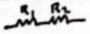


    MOSFET
- Inductor

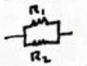
  $\phi = LI$

  $V = \frac{d\phi}{dt} = \frac{d}{dt}LI$ $I = I_0 e^{-t/(\frac{L}{R})}$

  ☆ $V = L\frac{dI}{dt}$ $E = \frac{1}{2}LI^2$
- Resistance Approximation



  $\max(R_1, R_2)$



  $\min(R_1, R_2)$

  max error = 2x

  - useful for Bode plots

  $\lambda = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$

  no oscillation = eigenvalues = real

  $R_{min} \geq 2\sqrt{\frac{L}{C}}$

- Phasors
  - Impedance

    $|Z| = \frac{|V|}{|I|}$
  - $z = |z|e^{j\theta}$ $|z| = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}(\frac{y}{x})$

    $z = x + jy$, $\bar{z} = x - jy$ (complex conjugate)
  - Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$

    $\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$ $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$
  - Trig Identities
    - $\sin x = \cos(x - \pi/2)$
    - $\cos x = \sin(x + \pi/2)$

    $*(-\cos x) = \cos(x \pm \pi)$
    - $\sin(-x) = -\sin x$
    - $\cos(-x) = \cos x$
  - Phasor
    - $V(t) = V_0\cos(\omega t + \phi_v)$

      $= \frac{V_0}{2}e^{j\phi}e^{j\omega t} + \frac{V_0}{2}e^{-j\phi}e^{-j\omega t}$
    - coefficient of $e^{j\omega t}$

      $\tilde{V} = \frac{V_0}{2}e^{j\phi}$
  - Impedance

    $Z_R = R$

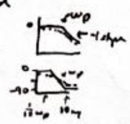
    $Z_C = \frac{1}{j\omega C}$

    $Z_L = j\omega L$
  - Transfer Function
    - $H(\omega) = |H(\omega)|e^{j\angle H(\omega)}$
    - $|a \cdot b| = |a||b|$
    - $|\frac{a}{b}| = \frac{|a|}{|b|}$
    - $\angle\frac{a}{b} = \angle a - \angle b$
  - Bode Plot
    - magnitude (log-log plot) $|H(\omega)|$

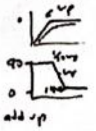
      split piecewise, approximate

      $\log_{10}(|H(\omega)|) = m\log_{10}(\omega) + k$
    - phase (semi-log) $\angle H(\omega)$
    - plotting - approximate
      - pole except ω/ω_p and zeroes
      - note error at ω_p
        - magnitude $\frac{1}{\sqrt{2}}$ at ω_p
      - phase
        - $\tan^{-1}(\frac{1}{10}) = 5.7°$
        - $\tan^{-1}(10) = 90 - 5.7°$

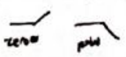
        we usually approximate then
    - magnitude/phase
      - LPF $\frac{1}{1 + j\omega/\omega_p}$



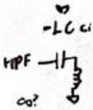
      - HPF $\frac{j\omega/\omega_p}{1 + j\omega/\omega_p}$



    - plot zeros and poles, add up

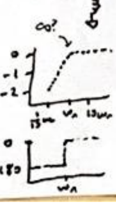


- LC circuits



  $H(\omega) = \frac{\omega^2 LC}{\omega^2 LC - 1}$ $\omega_n = \frac{1}{\sqrt{LC}}$

  $H(\omega) = \begin{cases} -\omega^2 LC & \omega \ll \omega_n \\ \infty & \omega = \omega_n \\ 1 & \omega \gg \omega_n \end{cases}$



- LC tank

  $V = L\frac{d}{dt}(-C\frac{dV}{dt})$

  $V = -LC\frac{d^2}{dt^2}V$

  $V + LC\frac{d^2}{dt^2}V = 0$

  $LC\omega^2 + 1 = 0$

  $\omega_n = \frac{1}{\sqrt{LC}}$

  natural oscillations

Matthew Tran

EE16B

Hola!!

hello

- Transient Style Analysis of Larger Circuits
  - $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$
    - Change of variables to eigenbasis
      - Remember $A = V\Lambda V^{-1}$
      - $\vec{x}(t) = V\vec{\tilde{x}}(t)$
      - $\frac{d}{dt}\vec{\tilde{x}}(t) = \tilde{A}\vec{\tilde{x}}(t)$
      - $\tilde{A} = V^{-1}AV$ (substitute into $\tilde{x}(t)$)
      - $\tilde{A} = V^{-1}V\Lambda V^{-1}V$

        $\tilde{A} = \Lambda$ ☺ woah
- Input in Diff Eq
  - $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$
  - soln from hw (only one) $x(t) = x_0 e^{\lambda t} + \int_0^t u(\tau)e^{\lambda(t-\tau)}d\tau$
  - Try $u(t) = H(s)e^{st}$, $s \neq \lambda$, $H(s) = \frac{1}{s-\lambda}$
    - soln $x(t) = (x_0 - H(s))e^{\lambda t} + H(s)e^{st}$
- Piecewise constant inputs

  (1) $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$

  (2) $u(t) = u(i)$ if $t \in [i\Delta, (i+1)\Delta)$

  Start at $x_d[i] = x(i\Delta) \Rightarrow x_d[i+1] = x((i+1)\Delta)$

  $\frac{d}{dt}x(t) = \lambda x(t) + u(i\Delta)$

  Guess $x(t) = \alpha e^{\lambda(t-i\Delta)} + \beta$

  $\frac{d}{dt}x(t) = \lambda\alpha e^{\lambda(t-i\Delta)} = \lambda x(t) + u(i\Delta)$

  $= \lambda(\alpha e^{\lambda(t-i\Delta)} + \beta) + u(i\Delta)$

  $\Rightarrow \lambda\alpha e^{\lambda(t-i\Delta)} = \lambda\alpha e^{\lambda(t-i\Delta)} \underbrace{+ \lambda\beta + u(i\Delta)}_{0}$

  $\beta = \frac{-u[i]}{\lambda}$, $x(i\Delta) = x[i]$

  $x[i] = \alpha + \beta \Rightarrow \alpha = x[i] + \frac{u[i]}{\lambda}$

  $x(t) = (x[i] + \frac{u[i]}{\lambda})e^{\lambda(t-i\Delta)} - \frac{u[i]}{\lambda}$

  $\Rightarrow x(i\Delta) = x[i]e^{\lambda(t-i\Delta)} + (\frac{e^{\lambda(t-i\Delta)} - 1}{\lambda})u[i]$

  $x[i+1] = x((i+1)\Delta) = x[i]e^{\lambda\Delta} + \frac{e^{\lambda\Delta} - 1}{\lambda}u[i]$
- Can we reach a certain position?
  - $\vec{x}(i+1) = A\vec{x}(i) + B\vec{u}(i) + \vec{w}(i)$

    past state, input, noise
  - $x[i] = A^i x[0] + B\sum_{j=0}^{i-1} u[j]A^{i-1-j}$
  - ex $\vec{x}(3) = A^3 x(0) + A^2 B\vec{u}(0) + AB\vec{u}(1) + B\vec{u}(2)$
  - to reach in 3 steps see if $x^* \in \underline{\text{span}(B, AB, A^2B)}$ (or $x^* - A^3x(0)$)?

    ↑ also keep in mind $A^n\vec{x}(0)$
- Finding A & B, least squares
  - $A\vec{x} = \vec{b}$ $\vec{x} = (A^TA)^{-1}A^T\vec{b}$, $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ Are constants

  - Find $\vec{x}(0)$?
    - observation matrix $\vec{y}(i) = C\vec{x}(i)$ $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\vec{x}(0) = (\vec{y}(\rightarrow)$
      - consider system w/o inputs to get $\vec{x}(0)$

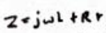
      when full rank, can solve
  - Assuming model scalar (parallel systems if not scalar)
    - want randomized $u(i)$ bc could have linear dependence
    - consider $x(i+1) = ax(i) + bu(i) + w(i)$ least squares $\begin{bmatrix} x(1) \\ x(2) \\ x(3) \\ \vdots \end{bmatrix} = \begin{bmatrix} x(1)u(1) \\ \vdots \\ x(n)u(n) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$
    - basically variable for each component
  - Stability?
    - Scalar 1) $x(i+1) = 0x(i) + u(i) + w(i)$ // no dependence, noise not.

      2) $x(i+1) = \frac{1}{2}x(i) + u(i) + w(i)$ // $\frac{1}{2} \cdot \frac{1}{2} \cdot \ldots = 0$ no effect

      3) $x(i+1) = 2x(i) + u(i) + w(i)$ // $2 \cdot 2 \cdot 2 \ldots = \infty$ unstable
    - Stable $|\lambda_i(\Delta)| < 1$
    - unstable $|\lambda_i(\Delta)| \geq 1$

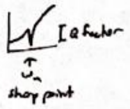
    Remember $A\vec{x}(i) = \lambda\vec{x}(i)$

    $A\vec{v} = \lambda\vec{v}$ $\det(A - \lambda I) = 0$
- Real inductor (has Ω)



  $Z = j\omega L + R + \frac{1}{j\omega C}$

  $Z(\omega_n = \frac{1}{\sqrt{LC}}) = R$



  $Q = \frac{\omega_n L}{R} = \frac{1}{\omega_n RC}$

- Controller Canonical Form (CCF)
  - $\tilde{A} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$, $\vec{z}(t) = \begin{bmatrix} z(t-(n-1)) \\ \vdots \\ z(t) \end{bmatrix}$
  - $z(t+1) = a_{n-1} z(t) + a_{n-2} z(t-1) + \ldots + a_0 z(t-(n-1)) + u(t)$
  - controllable system of form $\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$
  - Find T s.t. $\vec{z} = T\vec{x}$, $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$
    - $R_n = [\vec{b} \; A\vec{b} \; A^2\vec{b} \; \cdots \; A^{n-1}\vec{b}]$
    - $\vec{a} = R_n^{-1} A^n \vec{b}$ gives coefficients of characteristic poly of A
    - Find $\tilde{R}_n = [\vec{\tilde{b}} \; \tilde{A}\vec{\tilde{b}} \; \cdots \; \tilde{A}^{n-1}\vec{\tilde{b}}]$, again $\vec{a} = \tilde{R}_n^{-1} \tilde{A}^n \tilde{B}$
    - $T = \tilde{R}_n R_n^{-1}$ $\quad$ " $\begin{bmatrix} \vec{t}_1 & \cdots & \vec{t}_{n-1} \end{bmatrix}$
    - if use feedback $u(t) = \tilde{K}\vec{z} = K\vec{x}$, $\tilde{K} = KT^{-1}$, we get

      $T(A+BK)T^{-1} = \tilde{A} + \tilde{B}\tilde{K}$
  - Alternate conversion - coefficient matching
    - $\tilde{A}$ and A have same characteristic polynomial
    - check for controllability first
    - $\det(\tilde{A} - \lambda I) = \lambda^n - a_{n-1}\lambda^{n-1} - \ldots - a_0$ ☆☆☆

- Linearization
  - Linear F(x) properties iff
    - scaling $\forall \alpha (F(\alpha x) == \alpha F(x))$
    - superposition $f(x_1 + x_2) = f(x_1) + f(x_2)$
  - Linearization of Function $f(x)$
    - expand about x^*
    - $f_L(x^* + \delta x) \overset{\Delta}{=} f(x^*) + m\delta x$
    - $\delta y = f_L(x^* + \delta x) - f(x^*) = m\delta x$ $\Big\} \left.\frac{df}{dx}\right|_{x^*}$
  - Linearity for System
    - $\xrightarrow[u(t)]{\text{input}} \boxed{\mathcal{L}\{\}} \xrightarrow[y(t)]{\text{output}}$ $\quad y(t) = \mathcal{L}\{u(t)\}$ ← functional

      not same as $y(t) \neq \mathcal{L}(u(t))$ ← function
    - function
      - easy, if know t, can get y(t) if know u(t)
    - functional (capture memory)
      - need to know all u(t)
      - ex $y(t) = \mathcal{L}\{u(t)\} \overset{\Delta}{=} \int_{t-1}^{t} u(\tau)\,d\tau$
    - system linear iff
      - scaling $\mathcal{L}\{\alpha u(t)\} = \alpha\mathcal{L}\{u(t)\}$
      - superposition $\mathcal{L}\{u_1(t) + u_2(t)\} = \mathcal{L}\{u_1(t)\} + \mathcal{L}\{u_2(t)\}$
    - ex $\frac{dx}{dt} = f(x) = bu(t)$ is linear iff $f(x)$ is linear
  - Linearization of ODE about $\overset{DC}{\text{operating}}$ point
    - $\frac{d}{dt} x = f(x) + bu(t)$, but $f(x)$ nonlinear

      1) choose/given input $u(t) = u^*$ // constant w/ time

      2) $x(t) = x^*$ Solve $f(x^*) = -bu^*$ for x^*

      3) Define $\delta x(t) = x(t) - x^*$, $\delta u(t) = u(t) - u^*$
         ↑ output deviation ↑ input perturbation

         $\frac{d}{dt}\delta x(t) = f(x^* + \delta x(t)) + bu^* + b\delta u(t)$

      4) Suppose $\delta u(t)$ is small

      5) Assume $\delta x(t)$ is small (big assumption, not always true)

      6) $\frac{d}{dt}\delta x(t) = f(x^*) + m\delta x(t) + bu^* + b\delta u(t)$

         $\frac{d}{dt}\delta x(t) = m\delta x(t) + b\delta u(t)$ ★
  - Linearization of Vector Case
    - $\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}_1, \vec{x}_2) \\ f_2(x_1, x_2) \end{bmatrix}$
    - $\vec{f}(x^* + \delta x) \approx \vec{f}(x^*) + J_{\vec{x}}\,\delta\vec{x}$ at $\vec{x}^*$
- Speed up OMP
  1) Find max column, add but orthonormalize
  2) $\vec{x} = (A^TA)^{-1}A^T\vec{b} = A^T\vec{b}$!
  3) keep going
- Gram Schmidt $\quad \text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \cdot \vec{a}$
  - don't forget to normalize too $\quad$ ↑ unit vector $\vec{a}$
- BIBO - bounded input, bounded output

- Schur Form - $A = QTQ^*$
  - ↑ upper triangular
  - unitary - $Q^*Q = I$
  - if real, orthonormal
  - basically find λ and Q is orthonormal basis with $\vec{v}$
- Spectral Theorem
  - real, symmetric $A = P\Lambda P^T$
    - ↑ unitary
- SVD (Singular Value Decomposition)
  - $A = U \Sigma V^T$
    - A: real n×m
    - U: real n×n unitary
    - Σ: n×m $\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \\ & 0 & \end{bmatrix}$, $\sigma_1 \geq \ldots \geq \sigma_m \geq 0$
    - V^T: m×m unitary, real
  - outer products
  - Moore Penrose Pseudo Inverse
    - $A^\dagger = V\tilde{\Sigma}U^T$
    - solves $A\vec{x} = \vec{b}$ w/ min $\|\vec{x}\|$
    - $\tilde{\Sigma} = \begin{bmatrix} \frac{1}{\sigma_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_m} \end{bmatrix}$ pad 0s where needed
    - good for min energy
  - Calculation Frobenius Norm

    $\|A\|_F = \sqrt{\sum_i \sum_j |a_{ij}|^2}$

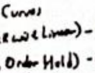
    $= \sqrt{\text{Tr}\{A^TA\}}$

    $= \sqrt{\sum_{i=1}^{n} \sigma_i^2}$
  - $A = U\Sigma V^T = \sum_{i=1}^{m} \sigma_i \vec{u}_i \vec{v}_i^T$
  - calculation - A - n×m
    1) A^TA dim n×n
    2) Find λ_i's and $\vec{v}_i$'s
    3) Find $\sigma_i = \sqrt{\lambda_i}$ sort by size, make V matching ↖ if not enough, use Gram Schmidt
    4) $\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}$
  - for compression error

    $\|A - \hat{A}\| = \sqrt{\sum_{i=k+1}^{m} \sigma_i^2}$
- PCA (Principal Component Analysis)
  - A - n×m
  - $\tilde{A}$ - mean centered
  - Covariance Matrix

    $S \overset{\Delta}{=} \frac{1}{n}\tilde{A}^T\tilde{A}$ m×m - diagonal has biggest vals absolute value-wise
  - Correlation Matrix
    - divide by s_i and s_j
    - diagonal 1s
    - r_{ij} is correlation between dims
    - not always good tho
  - PCA - $S = P\Lambda P^T$
    - $\vec{p}_i$ give direction of correlation
    - λ_i is covariance
  - SVD connection
    - $\tilde{A} = U\Sigma V^T \Rightarrow S = V\Sigma^TU^TU\Sigma V^T \cdot \frac{1}{n}$

      $= V\begin{bmatrix} \frac{\sigma_1^2}{n} & & \\ & \ddots & \\ & & \frac{\sigma_m^2}{n} \end{bmatrix}V^T$

      $= P\Lambda P^T \Rightarrow \lambda_i = \frac{\sigma_i^2}{n}$
- Approximation Curves
  - PWL (Piecewise Linear) - lines
  - ZOH (Zero Order Hold) - constant between points, jumps
  - ~~$\phi(t)$~~ - basis func()  $y_i\,\phi(t - i\Delta)$
  - sinc() $\text{sinc}(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{\sin(\pi t)}{\pi t} & \text{o/w} \end{cases}$ 
  - Polynomial

- Hermitian matrix - same as complex transpose conjugate
- To find closed loop λ w/ observation matrix C, just plug in $C\vec{\hat{x}}$ instead of $\vec{x}$ to get new char poly
- ~~ϕ~~ $v[t] = \theta u[t] - \beta$
  car

  $\begin{bmatrix} | & 1 \\ \vec{u}(t) & \vdots \\ | & 1 \end{bmatrix}\begin{bmatrix} \theta \\ -\beta \end{bmatrix} = \begin{bmatrix} | \\ \vec{v}(t) \\ | \end{bmatrix}$

- Discrete Fourier Transform (DFT)
  - Digital Signal Processing (DSP)
    - take snippets of signal and process
    - $x(t) = \underset{\text{DC}}{\overset{\uparrow}{A_0}} + \underbrace{A_1\cos(2\pi \overset{\text{fundamental freq}}{f_0} t + \theta_1)}_{\text{fundamental component}} + \underbrace{A_2\cos(2\pi \overset{\text{2nd harmonic freq}}{2f_0} t + \theta_2)}_{\text{2nd harmonic}} + \ldots$ chunk
    - phasors - $A_1\cos(2\pi f_0 t + \theta_1) = \frac{A_1}{2}e^{j\theta_1}e^{j2\pi f_0 t} + c.c.$
  - Complex review
    - $j = \sqrt{-1}$; $a = a_r + ja_i$; $\bar{a} = a_r - ja_i$
    - $a\bar{a} = a_r^2 + a_i^2$; $|a| = \sqrt{a_r^2 + a_i^2} = \sqrt{a\bar{a}}$
    - deMoivre's/Euler's formula - $e^{j\theta} = \cos\theta + j\sin\theta$
      - $e^{-j\theta} = \cos\theta - j\sin\theta$ // graphically true
      - $\cos\theta =$ ~~[illegible]~~ $\frac{1}{2}(e^{j\theta} + e^{-j\theta})$
      - $\sin\theta = \frac{1}{2}(e^{j\theta} - e^{j\theta})$
    - Polar Form - $a = a_r + ja_i = Me^{j\theta}$ $M = |a|$, $\theta = \tan^{-1}(\frac{a_i}{a_r})$
      - $x^k = M^k e^{jk\theta}$
      - roots of unity (1) - $\omega_n = e^{j\frac{2\pi}{n}} \Rightarrow \omega_n^n = 1 \Rightarrow \omega_n = \sqrt[n]{1}$
  - complex vectors/matrices
    - $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\|\vec{x}\| = \sqrt{\|x_1\|^2 + \ldots + \|x_n\|^2} = \langle\vec{x},\vec{x}\rangle$
    - $\langle\vec{x},\vec{y}\rangle \overset{\Delta}{=} \overline{\vec{y}^T}\vec{x}$ //conjugate ensures real
    - $y^* = y^H = \overline{y^T}$ Hermitian = complex conjugate transpose
  - DFT matrix
    - $N \times N$, indices start at 0, F_N
    - $F_N = \begin{array}{c|ccc} & 0 & 1 \; 2 \ldots & n-1 \\ \hline 0 & \omega_n^{-0\cdot 0} & \cdots & \omega_n^{-(n-1)\cdot 0} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & \omega_n^{-0(n-1)} & \cdots & \omega_n^{-(n-1)(n-1)} \end{array}$ $\omega_n^{-ij} = (i,j)^{th}$ entry
  - $F_N = \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$ $\vec{u}_k = \begin{bmatrix} \omega_N^{-0\cdot k} \\ \omega_N^{-1\cdot k} \\ \vdots \\ \omega_N^{-(N-1)k} \end{bmatrix}$
  - Properties
    1) symmetric
    2) $\|\vec{u}_k\| = \sqrt{N}$, bc $\|\vec{\omega}_n\| = 1$
    3) $\langle\vec{u}_k,\vec{u}_i\rangle = 0$ if $k \neq i$, orthogonal
    4) $F_N^{-1} = \frac{1}{N}F_N^* \leftrightarrow F_N^* F_N = NI$
    5) $\vec{u}_k = \overline{\vec{u}_{N-k}}$ $F_N = \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$ complex conjugate

       $\vec{X} = F_N \underset{\text{real}}{\underset{\uparrow}{\vec{x}}} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$ $\boxed{\exists}$ complex conjugate $\vec{X}_k = \vec{X}_{N-k}$
  - Frequency separation
    - $x(t) = A_\ell\cos(2\pi\ell f_0 + \theta_\ell)$, $T = \frac{1}{f_0}$, $t \in [0,T]$
    - Mf_0 is max considered frequency

      $N = 2M+1$ samples

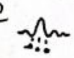
      $0 \;\; \Delta = \frac{T}{N} \qquad \overset{\uparrow}{(N-1)\Delta} \;\; T$
    - $x(k\Delta) = A_\ell\cos(2\pi\ell f_0 k\Delta + \theta_\ell) = A_\ell\cos(\frac{2\pi}{N}\ell k + \theta_\ell)$

      $= \frac{A_\ell}{2}e^{j\theta_\ell}\underbrace{e^{j\frac{2\pi}{N}\ell k}}_{\omega_N^{\ell k}} + c.c. = c_\ell\,\omega_N^{\ell\cdot k}$
    - $\vec{x}_{\ell f_0} = \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix} = c_\ell \underset{\vec{u}_\ell}{\begin{bmatrix} \omega_N^{\ell 0} \\ \vdots \\ \omega_N^{\ell(n-1)} \end{bmatrix}} + \underset{\text{contains } \vec{u}_\ell}{\overset{\uparrow}{c.c.}}$
      - $F_N\vec{x}_\ell = N\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$ 1 when matched and peak at c.c. ℓ & $N-\ell$
  - Periodic waveforms
    - T-periodic if $x(t+T) = x(t)$ for all t
    - $f_0 = \frac{1}{T}$ fundamental freq
    - make any T-periodic w/ apply function $p(t) = f(y(t))$
    - Fourier series - can make any signal w/ sine waves
    - $x(t) = \sum_{i=0}^{\infty} B_i\cos(2\pi i f_0 t + \theta_i) = \sum_{\ell=-\infty}^{\infty} A_\ell e^{j2\pi f_0 \ell t}$
    - $A_\ell = \frac{1}{T}\int_0^T e^{-j2\pi f_0 \ell t}\,x(t)\,dt$, each integer ℓ
    - $N = 2M+1$ samples, $f_0 = \frac{1}{T}$ $x(t) = \sum_{i=-M}^{M} \underset{\text{DFT, kinda } \begin{bmatrix} -M \\ \vdots \\ 0 \\ \vdots \\ M \end{bmatrix}}{\underset{\uparrow}{X_i}} e^{j\frac{2\pi}{T}it}$ $\vec{x} = F_N^* \underset{\text{reord}}{\underset{\uparrow}{\vec{X}}} = NF_N^{-1}\vec{X}$

- Interpolation by basis functions
  - $y(t) = \sum_i \alpha_i\phi_i(t)$; $\phi_i(t)$ continuous time func
  - $y(t) = \sum_{k=0}^{N-1} y_d(k)\phi(t - k\Delta)$
    - need $\phi(0) = 1$ and $\phi(k\Delta) = 0$ for $k \neq 0$
  - Zero Order Hold (ZOH) -

  - Piecewise Linear (PWL) -

  - $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$



$\phi(t) = \text{sinc}(\frac{t}{\Delta})$

- Interpolation by Global Polynomials
  - Lagrange Interpolation
  - $y(t) = \sum_{i=0}^{N-1} a_i t^i$, need $y(k\Delta) = y_d(k)$
    - $N-1$ equations - $y_d(k) = a_0 + a_1(k\Delta) + \ldots + a_{n-1}(k\Delta)^{n-1}$
    - $\begin{bmatrix} y_d(0) \\ \vdots \\ y_d(n-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \Delta & \Delta^2 \cdots & \Delta^{n-1} \\ \vdots & 2\Delta & \ddots & \vdots \\ 1 & (n-1)\Delta & \cdots & ((n-1)\Delta)^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix}$

      $\vec{y} = V\vec{a}$
    - Vandermonde structure - $\det(V) = (N-1)!\,\Delta \neq 0$
      - always a unique soln
    - Lagrange Interpolation
      - $L_i(x_i) = 1$; $L_i(x) = \prod_{j=0, j\neq i}^{j=n-1} \frac{(x - x_j)}{(x_i - x_j)}$
      - $y(t) = \sum_{i=0}^{n-1} y_d(i)L_i(t)$

- Extra
  - LTI - linear time invariant system
    - $y[n] = \underset{\text{impulse}}{h[n]} \underset{\text{convolution}}{*} \underset{\text{input}}{x[n]}$
    - BIBO - bounded input, bounded output
  - DFT coeffs $\frac{N}{2}e^{j\theta}$ $\cos(\frac{2\pi\ell t}{N} + \theta)$
    - $= \frac{NA_\ell}{2}e^{j\theta}$
  - $\frac{d}{dt}\vec{x}(t) = \Lambda\vec{x}(t)$
    - if eigenvalue real and negative $\Rightarrow$ stable
    - oscillates if imaginary part
    - stable if negative eigenvalue
  - k-means
    - classify and recalculate
    - clusters
  - SVD good for pseudoinverse
    - soln for system w/ min norm
  - DFT - think of it like counting number of waves in sample window